SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #2

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 92

- Attempt questions 1 – 6

Examiner: *C.Kourtesis*

Section A
(Start a new answer sheet.)

Question 1. (18 marks)

Marks

(a) If $z = 1 + 2i$ and $w = 3 - i$ express $z^2 \div \overline{w}$ in the form $a + ib$ (where a and b are real). **3**

(b) Sketch the region in the Argand diagram represented by $|z + i| \le 2$. **2**

(c) Find all pairs of integers a and b that satisfy $(a + ib)^2 = -3 - 4i$. **3**

(d) If $|z - 1| = \text{Re}(z) + 1$ find the locus of z. **3**

(e) If z and w are two complex numbers, prove that **3**

$$\overline{z - w} = \overline{z} - \overline{w}.$$

(f) (i) Express each of the complex numbers $z = 2i$ and $w = 1 + i\sqrt{3}$ in modulus-argument form. **2**

(ii) Find the exact value of $\arg(z + w)$. **2**

Question 2. (15 marks)

Marks

(a) Find $\int \frac{dx}{x^2+2x+5}$. **2**

(b) Find $\int \frac{\sin x\, dx}{(2+7\cos x)^5}$ using the substitution $u = 2+7\cos x$. **2**

(c) Use integration by parts to find: **3**

$$\int x^3 \ln x\, dx$$

(d) (i) Find real numbers a, b, and c such that **2**

$$\frac{6}{x^2(x+3)} = \frac{ax+b}{x^2} + \frac{c}{x+3}.$$

(ii) Find $\int \frac{6}{x^2(x+3)}\, dx$ **2**

(e) Use the substitution $t = \tan \frac{x}{2}$ to evaluate **4**

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\, dx$$

Give your answer in simplest exact form.

Section B
(Start a new answer sheet.)

Question 3. (14 marks)

Marks

(a) Find the equation whose roots are twice those of the equation: **2**

$$x^3 - x - 3 = 0.$$

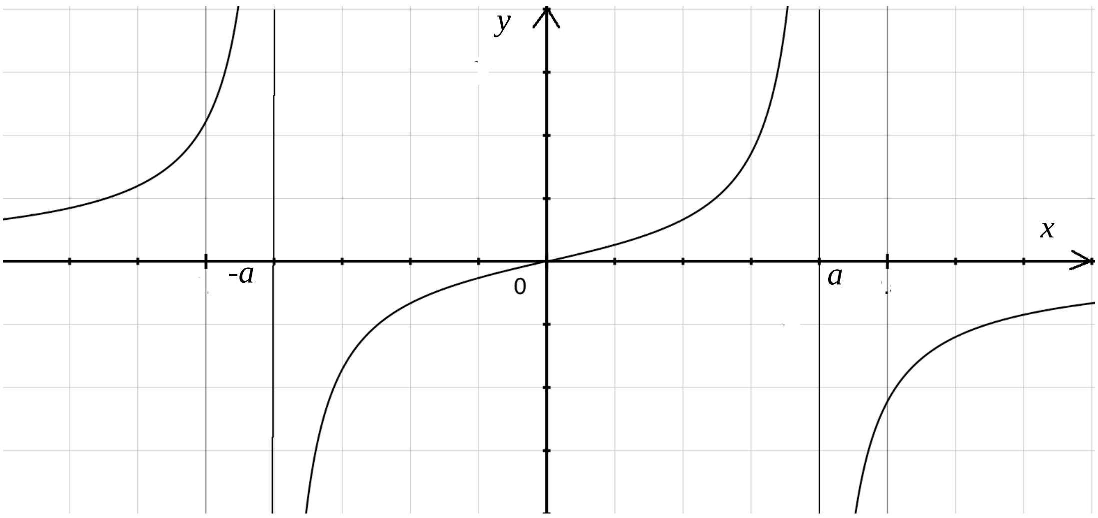
(b) (i) Prove that if α is a double root of the equation $P(x) = 0$ then $P'(\alpha) = 0$. **2**

(ii) Prove that $P(x) = x^3 - 3ax + b$ has a double root if $b^2 = 4a^3$. **3**

(iii) Hence or otherwise solve the equation **2**

$$x^3 + 3x + 2i = 0$$

(c) The graph of $y = f(x)$ is shown:



Draw sketches of the following curves (on separate diagrams):

(i) $y = [f(x)]^2$ **1**

(ii) $y = f'(x)$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

Question 4 (15 marks)

(a) Sketch the graphs of:

(i) $y = \sin^2 2x$ for $-2\pi \le x \le 2\pi$. **2**

(ii) $\sin(x + y) = 0$ **2**

(b) (i) Sketch the graph of $y = |x-1|(x+1)$. **2**

(ii) Hence or otherwise sketch the graph of **2**

$$y = \frac{1}{|x-1|(x+1)}.$$

(c) Consider the function:

$$f(x) = \frac{e^{-x} - e^{x}}{e^{x} + e^{-x}}$$

(i) Show that *f(x)* is an odd function. **1**

(ii) Find the equations of any asymptotes. **2**

(iii) Show that *f(x)* is a decreasing function for all real *x*. **2**

(iv) Sketch the graph of *y* = *f(x)*. **2**

Section C
(Start a new answer booklet)

Question 5 (13 marks)

(a) Sketch the graph of the function **4**

$$y = x^{\frac{1}{3}} + \tfrac{1}{4}x^{\frac{4}{3}}$$

indicating the nature of any turning points and the co-ordinates of any points of inflexion.

(b) A rectangle is divided by *m* lines parallel to one pair of opposite sides and by *n* lines parallel to the other pair of opposite sides. **2**

How many rectangles of any size are formed in the resulting figure?
(Leave your answer in unsimplified form.)

(c) Given that $z_1 = f(z) = \dfrac{z+i}{z-i}$, show that $f(z_1) = \left(\dfrac{z+1}{z-1}\right).\, i$ **3**

(d) If *a* and *b* are two roots of the equation $x^3 + 4x - 2 = 0$, show that *ab* is a root of the equation $x^3 - 4x^2 - 4 = 0$. **4**

Question 6 (17 marks)

(a) Consider the function

$$f(x) = x + \log_e(1 - x)$$

(i) Sketch the graph of $y = f(x)$ showing all essential features. **3**

(ii) Hence show that $x \le -\log_e(1 - x)$ for all $x < 1$. **1**

(b)

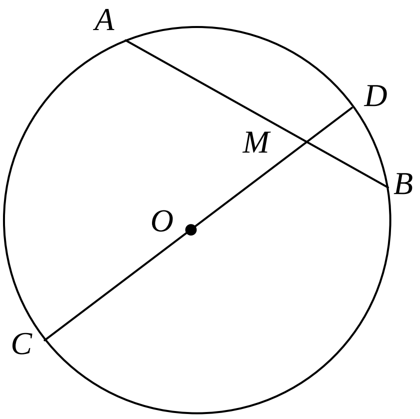
(i) Show that $\int_0^{\frac{\pi}{2}} x\cos x\,dx = \frac{\pi}{2} - 1$. **2**

(ii) If $u_n = \int_0^{\frac{\pi}{2}} x\cos^n x\,dx$ and n is a positive integer greater than 1, prove that **4**

$$u_n = \left(\frac{n-1}{n}\right)u_{n-2} - \frac{1}{n^2}.$$

(iii) Deduce that $u_5 = \frac{4\pi}{15} - \frac{149}{225}$. **2**

(c)



A chord AB and a diameter CD of a circle centre O intersect at a point M within the circle.
(M is not the centre.)

(i) Show that $(CM + MD)^2 > (AM + MB)^2$ **2**

(ii) Deduce that $(CM - MD)^2 > (AM - MB)^2$ **3**

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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Sample Solutions

Section	Marker
A	PSP
B	PRB
C	DMH

Section A

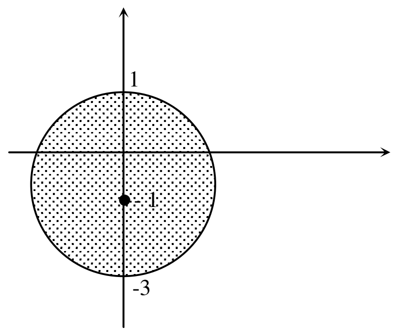
Question 1

(a) $z = 1 + 2i, w = 3 - i$

$\therefore z^2 = 1 - 4 + 4i = -3 + 4i, \overline{w} = 3 + i$

$\therefore \dfrac{z^2}{\overline{w}} = \dfrac{-3+4i}{3+i} \times \dfrac{3-i}{3-i} = \dfrac{-5+15i}{10} = -\dfrac{1}{2} + \dfrac{3}{2}i$

(b) The interior of a circle centred at $z = -i$ and radius of 2 units.



(c) $z = a + ib$

$z^2 = (a+ib)^2 = -3 - 4i$

$\therefore a^2 - b^2 = -3,$

$2ab = -4 \Rightarrow ab = -2$

$|z^2| = 5 \Rightarrow |z|^2 = 5 \Rightarrow a^2 + b^2 = 5$

$\left.\begin{array}{l} a^2 + b^2 = 5 \\ a^2 - b^2 = -3 \end{array}\right\} +$

$\therefore 2a^2 = 2 \Rightarrow a = \pm 1$

$\therefore 2b^2 = 8 \Rightarrow b = \pm 2$

$\because ab < 0 \Rightarrow z = \pm(1 - 2i)$

$\therefore (1, -2) \,\&\, (-1, 2)$

(d) $|z - 1| = \mathrm{Re}(z) + 1$

$z = x + iy$

$\therefore |(x-1) + iy| = x + 1 \Rightarrow x \geq -1$

$\therefore \sqrt{(x-1)^2 + y^2} = x + 1$

$\therefore (x-1)^2 + y^2 = (x+1)^2$

$\therefore y^2 = (x+1)^2 - (x-1)^2 = (x+1-x+1)(x+1+x-1)$

$\therefore y^2 = (2)(2x) = 4x$

$\therefore y^2 = 4x$

(e) $z = z_1 + iz_2, w = w_1 + iw_2$

$\therefore z - w = (z_1 - w_1) + i(z_2 - w_2)$

$\therefore \overline{z - w} = (z_1 - w_1) - i(z_2 - w_2)$

$\overline{z} - \overline{w} = (z_1 - iz_2) - (w_1 - iw_2) = (z_1 - w_1) - i(z_2 - w_2)$

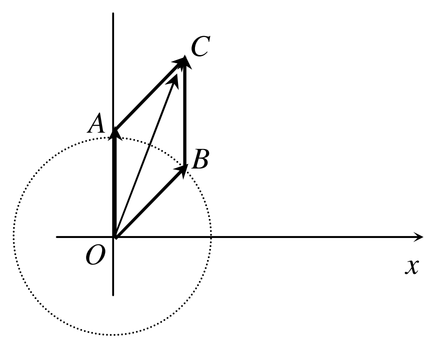
$\therefore \overline{z - w} = \overline{z} - \overline{w}$

(f) (i) $|2i| = 2, \arg(i) = \dfrac{\pi}{2} \Rightarrow z = 2\text{cis}\dfrac{\pi}{2}$

$\left|1 + i\sqrt{3}\right| = 2, \arg\left(1 + i\sqrt{3}\right) = \dfrac{\pi}{3} \Rightarrow w = 2\text{cis}\dfrac{\pi}{3}$

(ii) If C corresponds to the complex number $z + w$ then $OABC$ is a rhombus.

So $\angle COA = \frac{1}{2}\angle AOB = \dfrac{\pi}{12}$ [properties of rhombi]



$$\begin{aligned}\arg(z + w) &= \angle COx \\ &= \angle BOx + \tfrac{1}{2}(\angle AOB) \\ &= \frac{\pi}{3} + \tfrac{1}{2}\left(\frac{\pi}{6}\right) \\ &= \frac{5\pi}{12}\end{aligned}$$

Question 2

(a) $$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$$
$$= \frac{1}{2}\tan\left(\frac{x+1}{2}\right)+c$$

(b) $u = 2+7\cos x$

$du = -7\sin x dx$

$$\int \frac{\sin x dx}{(2+7\cos x)^5} = -\frac{1}{7}\int \frac{-7\sin x dx}{(2+7\cos x)^5}$$
$$= -\frac{1}{7}\int \frac{du}{u^5}$$
$$= -\frac{1}{7}\int u^{-5} du$$
$$= -\frac{1}{7}\left[-\frac{1}{4}u^{-4}\right]+c$$
$$= \frac{1}{28(2+7\cos x)^4}+c$$

(c) $$\int \underbrace{x^3}_{f'} \underbrace{\ln x}_{g} dx = fg - \int fg' dx$$
$$= \frac{1}{4}x^4 \ln x - \int \left(\frac{1}{4}x^4 \times \frac{1}{x}\right) dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$$

(d) (i) $$\frac{6}{x^2(x+3)} = \frac{ax+b}{x^2} + \frac{c}{x+3}$$

$\therefore 6 = (ax+b)(x+3)+cx^2$

sub $x=-3: 6 = c(9) \Rightarrow c = \frac{2}{3}$

$a+c=0 \Rightarrow a = -\frac{2}{3}$ [coefficient of x^2]

$3b = 6 \Rightarrow b = 2$ [constant term]

$\therefore a = -\frac{2}{3}, b = 2, c = \frac{2}{3}$

(ii) $$\frac{6}{x^2(x+3)}=\frac{-\frac{2}{3}x+2}{x^2}+\frac{\frac{2}{3}}{x+3}$$
$$=-\frac{2}{3}\left(\frac{x-3}{x^2}-\frac{1}{x+3}\right)$$
$$=-\frac{2}{3}\left(\frac{1}{x}-3x^{-2}-\frac{1}{x+3}\right)$$

$$\int \frac{6}{x^2(x+3)}dx=-\frac{2}{3}\int\left(\frac{1}{x}-3x^{-2}-\frac{1}{x+3}\right)dx$$
$$=-\frac{2}{3}\left[\ln x+3x^{-1}-\ln(x+3)\right]+c$$
$$=-\frac{2}{3}\ln\left(\frac{x}{x+3}\right)-\frac{2}{x}+c$$
$$=\frac{2}{3}\ln\left(\frac{x+3}{x}\right)-\frac{2}{x}+c$$

(e) $$t=\tan\left(\frac{x}{2}\right)\Rightarrow x=2\tan^{-1}(t)$$
$$\therefore dx=\frac{2}{1+t^2}dt$$
$$x=0\Rightarrow t=0$$
$$x=\frac{\pi}{3}\Rightarrow t=\frac{1}{\sqrt{3}}$$
$$\sin x=\frac{2t}{1+t^2}\Rightarrow 1-\sin x=\frac{1+t^2-2t}{1+t^2}=\frac{(t-1)^2}{1+t^2}$$

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx = \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{1}{\frac{(t-1)^2}{1+t^2}}\right) \times \frac{2dt}{1+t^2} = \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{(t-1)^2}$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} (t-1)^{-2}\,dt$$

$$= -2\left[(t-1)^{-1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= -2\left[\frac{1}{t-1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= -2\left[\frac{1}{\frac{1}{\sqrt{3}}-1} - (-1)\right]$$

$$= -2\left[\frac{\sqrt{3}}{1-\sqrt{3}} + 1\right]$$

$$= \frac{2\sqrt{3}}{\sqrt{3}-1} - 2$$

$$= \frac{2}{\sqrt{3}-1}$$

$$= \sqrt{3}+1$$

Section B

QUESTION 3

a Given $x^3 - x - 3 = 0$ — (A)

Form a new equation where $X = 2x \Rightarrow x = \frac{X}{2}$ in (A)

$(\frac{X}{2})^3 - \frac{X}{2} - 3 = 0$

$\frac{X^3}{8} - \frac{X}{2} - 3 = 0$

$\boxed{X^3 - 4X - 24 = 0}$

b (i) Let $P(x) = (x - \alpha)^2 . \varphi(x)$.

$\therefore \quad P'(x) = 2(x - \alpha) . \varphi(x) + (x - \alpha)^2 . \varphi'(x)$

$\therefore P'(\alpha) = 2(\alpha - \alpha) . \varphi(\alpha) + (\alpha - \alpha)^2 . \varphi'(\alpha)$

$= 2 \times 0 \times \varphi(\alpha) + 0 \times \varphi'(\alpha)$

$= 0$.

(ii) Given $P(x) = x^3 - 3ax + b$. — (A)

now $P'(x) = 3x^2 - 3a$ — (B)

If $P(x) = 0$ has a double root, say α.

then $P(\alpha) = P'(\alpha) = 0$.

From (B) if $P'(\alpha) = 0 \quad 3\alpha^2 - 3a = 0$

$\therefore \alpha^2 = a$.

In (A) $\quad \alpha^3 - 3\alpha a + b = 0$

$\alpha(\alpha^2 - 3a) \quad = -b$.

$\alpha(a - 3a) \quad = -b$

$\alpha \times -2a \quad = -b$.

$\alpha^2 \times 4a^2 \quad = b^2$

$\therefore a \times 4a^2 \quad = b^2$

$\boxed{4a^3 = b^2}$

(iii) Consider $x^3 + 3x + 2i = 0$

now $a = -1 \Rightarrow 4a^3 = -4$

and $b = 2i \Rightarrow b^2 = -4$

$\therefore$ a double root exists

Consider $3x^2 = 3a$

$x^2 = -1$

$x = \pm i.$

now $P(i) = -i + 3i + 2i \neq 0$

$\therefore \quad P(-i) = i - 3i + 2i = 0$

$\therefore \quad -i$ is the double root.

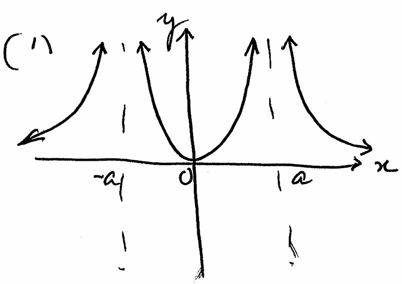
now $S_1 = \sum \alpha = 0$

$\therefore -i - i + \beta = 0$

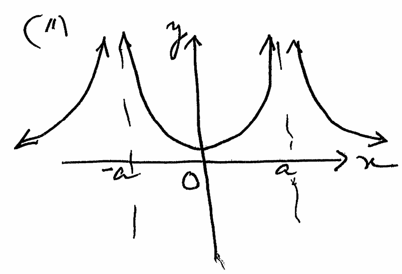
$\beta = 2i$

$\therefore$ roots are $\boxed{-i, -i \text{ \& } 2i}$.

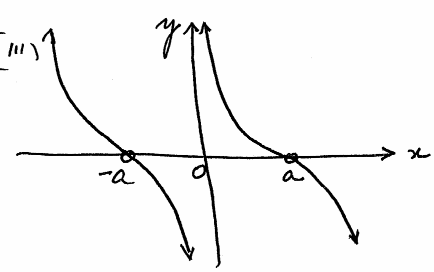
(c) (i)



(ii)

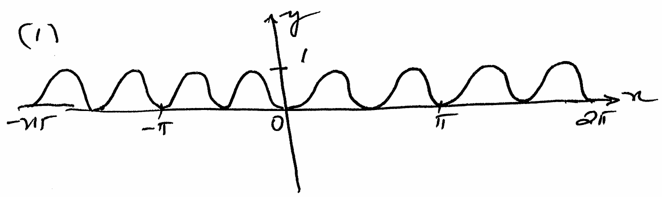


(iii)



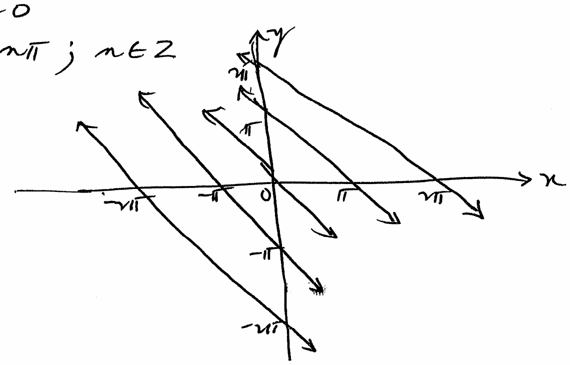
QUESTION 4

(a) (i)



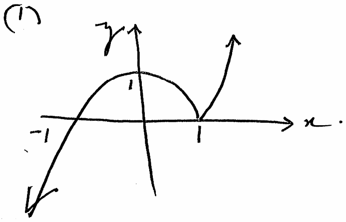
(ii) $\sin(x+y)=0$

$x+y=n\pi\ ;\ n\in\mathbb{Z}$

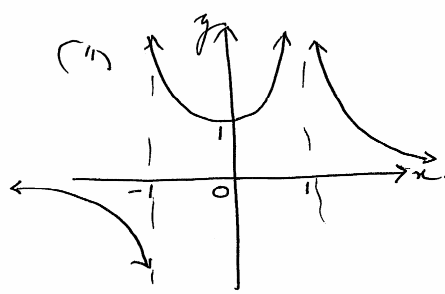


(b)

(i)



(ii)



(c) (i) $f(x)=\dfrac{e^{-x}-e^{x}}{e^{x}+e^{-x}}$

$f(-x)=\dfrac{e^{x}-e^{-x}}{e^{-x}+e^{x}}=-\dfrac{(e^{-x}-e^{x})}{e^{x}+e^{-x}}=-f(x)\ \therefore$ ODD!!

(ii) For vertical asymptotes

$e^{x}+e^{-x}=0$

$e^{2x}+1=0$

$e^{2x}=-1$ NO!!

$f(x)=\dfrac{e^{-2x}-1}{1+e^{-2x}}=\dfrac{1-e^{2x}}{e^{2x}+1}$

as $x\to\infty$ $f(x)\to\frac{-1}{1}=-1$

as $x\to-\infty$ $f(x)\to\frac{1}{1}=1.$

∴ horizontal asymptote $y = 1$ as $x \to -\infty$.

$y = -1$ as $x \to \infty$.

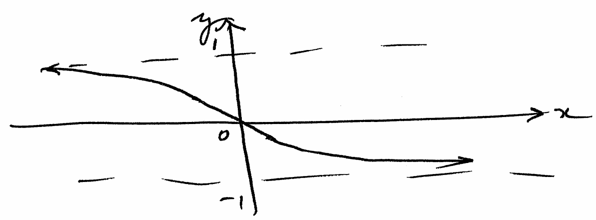
(iii) $$f'(x) = \frac{(e^{x}+e^{-x})(-e^{-x}-e^{x}) - (e^{-x}-e^{x})(e^{x}-e^{-x})}{(e^{x}+e^{-x})^2}$$

$$= \frac{-1-e^{2x}-e^{-2x}-1 \; -1+e^{-2x}+e^{2x}-1}{(e^{x}+e^{-x})^2}$$

$$= \frac{-4}{(e^{x}+e^{-x})^2}$$

< 0 ∴ decreasing.

(iv)



Section C

5. (a) Sketch the graph of the function

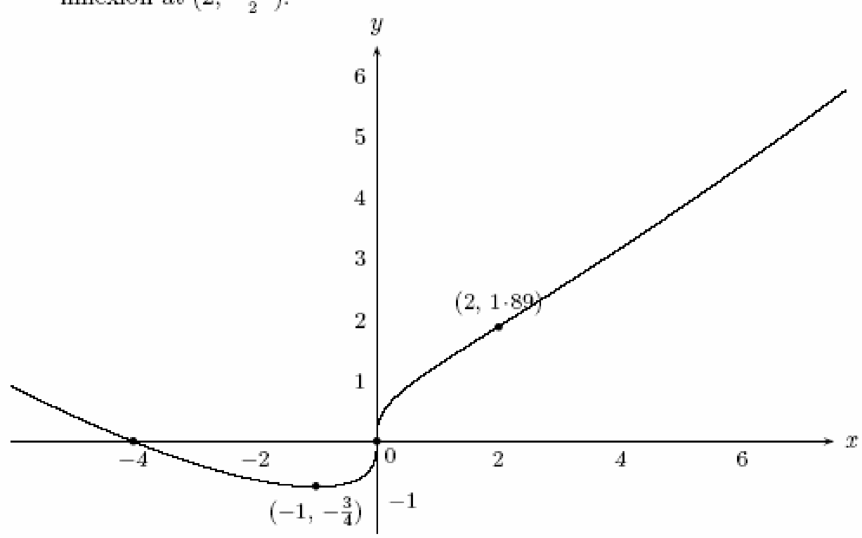
$$y = x^{\frac{1}{3}} + \tfrac{1}{4}x^{\frac{4}{3}}$$

indicating the nature of any turning points and the nature of any points of inflexion.

Solution:

$$\begin{aligned}
y &= \frac{\sqrt[3]{x}}{4}(4+x), \\
&= 0 \text{ when } x = 0,\ -4. \\
y' &= \tfrac{1}{3}x^{-\frac{2}{3}} + \tfrac{1}{3}x^{\frac{1}{3}}, \\
&= \frac{(1+x)}{3\sqrt[3]{x^2}}, \\
&= 0 \text{ when } x = -1, \\
&= \text{undefined when } x = 0. \\
y'' &= -\frac{2x^{-\frac{5}{3}}}{9} + \frac{x^{-\frac{2}{3}}}{9}, \\
&= \frac{x-2}{9x\sqrt[3]{x^2}}, \\
&= \tfrac{1}{48} \text{ when } x = 8, \\
&= 0 \text{ when } x = 2, \\
&= -\tfrac{1}{3} \text{ when } x = 1, \\
&= \tfrac{1}{3} \text{ when } x = -1.
\end{aligned}$$

$\therefore$ Minimum at $(-1,\ -\frac{3}{4})$,
vertical inflexion at $(0,\ 0)$,
inflexion at $(2,\ \frac{3\sqrt[3]{2}}{2})$.



(b) A rectangle is divided by m lines parallel to one pair of opposite sides and by n lines parallel to the other pair of opposite sides.

How many rectangles of any size are formed in the resulting figure? (Leave your answer in unsimplified form.)

Solution: $\binom{m+2}{2} \times \binom{n+2}{2}$

or ${}^{m+2}C_2 \times {}^{n+2}C_2$.

(c) Given that $z_1 = f(z) = \dfrac{z+i}{z-i}$, show that $f(z_1) = \left(\dfrac{z+1}{z-1}\right).i$

Solution:

$$\begin{aligned} f(z_1) &= \frac{\frac{z+i}{z-i}+i}{\frac{z+i}{z-i}-i}, \\ &= \frac{z+i+i(z-i)}{z+i-i(z-i)}, \\ &= \frac{z+i+iz+1}{z+i-iz-1}, \\ &= \frac{z+1+i(z+1)}{z-1-i(z-1)}, \\ &= \frac{(z+1)(1+i)}{(z-1)(1-i)} \times \frac{1+i}{1+i}, \\ &= \left(\frac{z+1}{z-1}\right)\left(\frac{1+2i-1}{1+1}\right), \\ &= \left(\frac{z+1}{z-1}\right).i \end{aligned}$$

(d) If a and b are two roots of the equation $x^3 + 4x - 2 = 0$, show that ab is a root of the equation $x^3 - 4x^2 - 4 = 0$.

Solution: Let c be the third root of $x^3 + 4x - 2 = 0$,

$$\begin{aligned} \text{then, } a+b+c &= 0 \quad \ldots \boxed{1} \\ ab+bc+ca &= 4 \quad \ldots \boxed{2} \\ abc &= 2 \quad \ldots \boxed{3} \end{aligned}$$

Method 1:

From $\boxed{1}$ $c = -(a+b) \quad \ldots \boxed{4}$

From $\boxed{3}$ $c = \dfrac{2}{ab} \quad \ldots \boxed{5}$

$$\begin{aligned} \text{Sub. } \boxed{5} \text{ in } \boxed{2},\ ab + \frac{2b}{ab} + \frac{2a}{ab} &= 4 \\ a^2b^2 + 2(a+b) &= 4ab \\ a^2b^2 - 2c &= 4ab \\ a^2b^2 - \frac{4}{ab} - 4ab &= 0 \\ (ab)^3 - 4(ab)^2 - 4 &= 0 \end{aligned}$$

i.e., ab is a root of $x^3 - 4x^2 - 4 = 0$.

Method 2: Consider the equation with roots ab, bc, ca.

Sum of roots one at a time is $ab + bc + ca = 4$.

Sum of roots two at a time is
$$\begin{aligned} ab^2c + bc^2a + ca^2b &= abc(a+b+c), \\ &= 0. \end{aligned}$$

Product of roots is
$$\begin{aligned} ab^2c^2a &= (abc)^2, \\ &= 4. \end{aligned}$$

$\therefore$ The equation is $x^3 - 4x^2 - 4 = 0$.

Method 3: $ab = \dfrac{2}{c}$ (from $\boxed{3}$ above).

Substitute into $x^3 - 4x^2 - 4 = 0$,

$$\begin{aligned} \text{L.H.S.} &= \frac{b}{c^3} - \frac{16}{c^2} - 4, \\ &= \frac{-4c^3 - 16c + 8}{c^3}, \\ &= \frac{-4}{c^3}\big(c^3 + 4c - 2\big) \end{aligned}$$

But c is a root of $x^3 + 4x - 2 = 0$,

$$\begin{aligned} \text{L.H.S.} &= 0, \\ &= \text{R.H.S.} \end{aligned}$$

$\therefore$ ab is a root of $x^3 - 4x^2 - 4 = 0$.

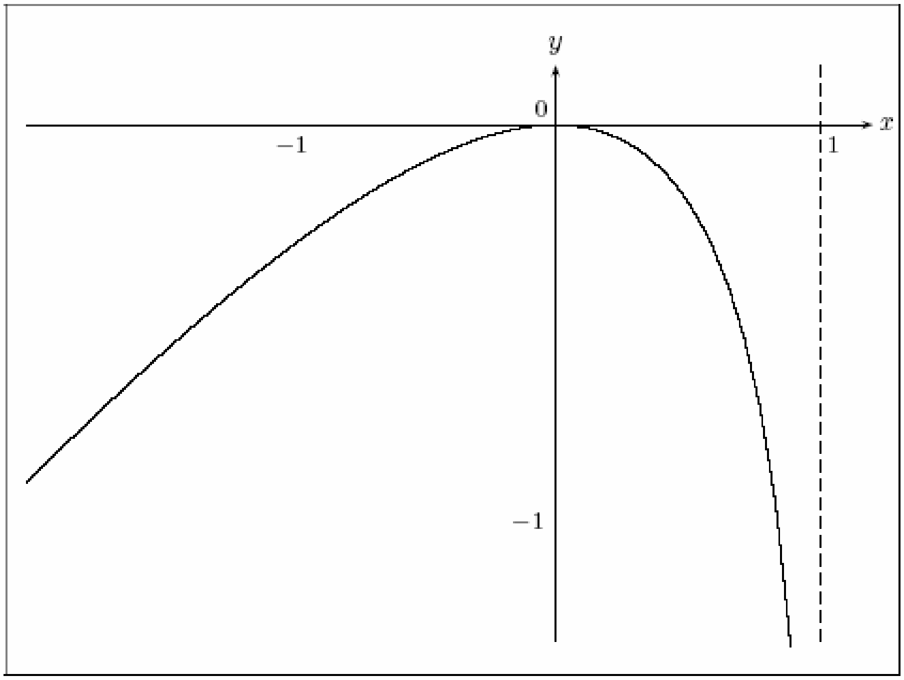
6. (a) Consider the function
$$f(x) = x + \log_e(1 - x)$$

i. Sketch the graph of $y = f(x)$, showing all essential features.

Solution:
$$\begin{aligned} f(x) &= x + \ln(1-x), \quad x < 1 \\ f(0) &= 0. \\ f'(x) &= 1 - \frac{1}{1-x}, \\ &= \frac{-x}{1-x} \\ &> 0 \text{ for } x < 0, \\ &= 0 \text{ for } x = 0, \\ &< 0 \text{ for } 0 < x < 1. \\ f'(1) &= \text{undefined.} \\ f''(x) &= \frac{-1}{(1-x)^2}, \\ &< 0 \text{ for all } x. \end{aligned}$$

Hence maximum at (0, 0), no inflexions.

Vertical asymptote at $x = 1$.



ii. Hence show that $x \le \log_e(1-x)$ for all $x \le 1$.

Solution: First interpretation—
Question should have read:

Hence show that $x \le -\log_e(1-x)$ for all $x \le 1$.

i.e., $x + \ln(1-x) \le 0$.
And it is clear from the sketch above that $f(x) \le 0, \quad x \le 1$.

. . . .

Second interpretation—
Let the question stand, *i.e.*, $x - \ln(1-x) \le 0$.
Now consider

$$\begin{aligned} y = f(x) &= x - \ln(1-x), \\ f(0) &= 0. \\ f'(x) &= 1 + \frac{1}{1-x}, \\ &> 0 \text{ for } x < 1, \\ &= \text{undefined when } x = 1. \end{aligned}$$

The curve is monotonic increasing and passes through (0, 0). Hence the statement is false.

(b) i. Show that $\int_0^{\frac{\pi}{2}} x\cos x\,dx = \frac{\pi}{2} - 1$.

Solution:

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} x\cos x\,dx, \\ &= x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx, \\ &= \frac{\pi}{2} - 0 - [-\cos x]_0^{\frac{\pi}{2}}, \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

$$u = x \quad v' = \cos x$$
$$u' = 1 \quad v = \sin x$$

ii. If $u_n = \int_0^{\frac{\pi}{2}} x\cos^n x\,dx$ and n is a positive integer, prove that

$$u_n = \left(\frac{n-1}{n}\right) u_{n-2} - \frac{1}{n^2}.$$

Solution:

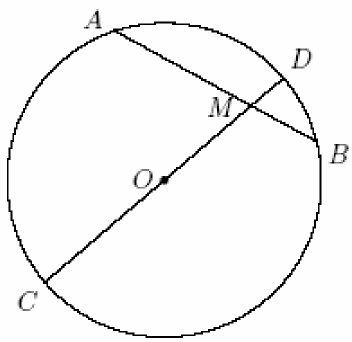
$$\begin{aligned} U_n &= \int_0^{\frac{\pi}{2}} x\cos^n x\,dx, \\ &= [x\sin x\cos^{n-1} x + \cos^n x]_0^{\frac{\pi}{2}} \\ &\quad + (n-1)\int_0^{\frac{\pi}{2}} x\sin^2 x\cos^{n-2} x\,dx \\ &\quad + (n-1)\int_0^{\frac{\pi}{2}} \sin x\cos^{n-1} x\,dx, \\ &= [x\sin x\cos^{n-1} x + \cos^n x]_0^{\frac{\pi}{2}} \\ &\quad + (n-1)\int_0^{\frac{\pi}{2}} x\cos^{n-2} x\,dx \\ &\quad - (n-1)\int_0^{\frac{\pi}{2}} x\cos^n x\,dx \\ &\quad - (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-1} x\,d\cos x, \\ &= \left[x\sin x\cos^{n-1} x + \cos^n x - \frac{(n-1)\cos^n x}{n}\right]_0^{\frac{\pi}{2}} \\ &\quad + (n-1)U_{n-2} - (n-1)U_n. \\ nU_n &= \left[x\sin x\cos^{n-1} x + \frac{\cos^n x}{n}\right]_0^{\frac{\pi}{2}} \\ &\quad + (n-1)U_{n-2}, \\ &= -\frac{1}{n} + (n-1)U_{n-2}. \\ \therefore\ U_n &= \frac{n-1}{n}U_{n-2} - \frac{1}{n^2}. \end{aligned}$$

$$\begin{aligned} u &= \cos^{n-1} x \\ v' &= x\cos x \\ u' &= (n-1)(-\sin x)\cos^{n-2} x \\ v &= x\sin x + \cos x \end{aligned}$$

iii. Deduce that $u_5 = \frac{4\pi}{15} - \frac{149}{225}$.

Solution:

$$U_5 = \tfrac{4}{5}U_3 - \tfrac{1}{25}.$$
$$U_3 = \tfrac{2}{3}U_1 - \tfrac{1}{9},$$
$$U_1 = \tfrac{\pi}{2} - 1.$$
$$\therefore\ U_5 = \frac{4}{5}\left(\frac{2}{3}\left\{\frac{\pi}{2} - 1\right\} - \frac{1}{9}\right) - \frac{1}{25},$$
$$= \frac{4\pi}{15} - \frac{8}{15} - \frac{4}{45} - \frac{1}{25},$$
$$= \frac{4\pi}{15} - \frac{149}{225}.$$

(c) 

A chord AB and a diameter CD of a circle centre O intersect at a point M within the circle.
(M is not the centre.)

i. Show that $(CM + MD)^2 > (AM + MB)^2$.

Solution: $CM + MD = CD$, a diameter.
$AM + MB = AB$, a chord which is not a diameter.
$CD > AB$, as a diameter is the longest chord in a circle.
$\therefore\ CD^2 > AB^2$.
So $(CM + MD)^2 > (AM + MB)^2$.

ii. Deduce that $(CM - MD)^2 > (AM - MB)^2$.

Solution:

$$(CM + MD)^2 > (AM + MB)^2,$$
$$CM^2 + 2CM.MD + MD^2 > AM^2 + 2AM.MB + MB^2.$$

Now $CM.MD = AM.MB$ (intersecting chord theorem),
so subtract $4CM.MD$ [$= 4AM.MB$] from both sides,

$$CM^2 - 2CM.MD + MD^2 > AM^2 - 2AM.MB + MB^2.$$
$$i.e.,\ (CM - MD)^2 > (AM - MB)^2.$$